

On the Fitting length of finite soluble groups I

Hall subgroups

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Abstract. *Let G be a finite soluble group and $h(G)$ its Fitting length. The aim of this paper is to give certain upper bounds for $h(G)$ as functions of the Fitting length of at least three Hall subgroups of G which factorize G in a particular way.*

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§1. Introduction.

If G is a group containing subgroups A and B such that

$$G = AB = \{ab \mid a \in A, b \in B\}$$

we say that G is factorized by A and B . A classical problem in group theory is, generally speaking, to try to get informations on the structure of G as a consequence of the structure of A and B . For example a celebrated theorem by Itô ([4]) states that any group $G = AB$ with A and B abelian is metabelian.

In this paper we only deal with finite soluble groups and so for us group will always mean “finite soluble group”. We shall consider the case when a group is factorized by Hall subgroups, namely subgroups whose order and index are coprime.

If G is a group, we shall denote by $d(G)$ and $h(G)$ respectively the derived length and the Fitting length of G , by $\pi(G)$ the set of prime divisors of $|G|$ and by $w(G)$ the cardinality of $\pi(G)$; also, we shall write π and w instead of $\pi(G)$ and $w(G)$ respectively when there is no possible ambiguity.

A recent result in [2] states that if $G = AB$, where A and B are Hall subgroups of G , then

$$h(G) \leq h(A) + h(B) + 4d(B) - 1.$$

Under the previous hypotheses it is not possible to give an upper bound for $h(G)$ as a function of $h(A)$ and $h(B)$ only; for, it is well known that there exist groups of arbitrary Fitting length which can be factorized by two Sylow subgroups. The aim of this paper is to prove that if A, B, C are Hall subgroups of a group G such that G is trifactorized by A, B, C , namely if

$$G = AB = BC = CA,$$

then there exists an upper bound for $h(G)$ as a function of $h(A)$, $h(B)$ and $h(C)$. Moreover, if G is k -factorized by Hall subgroups ($k \geq 3$; the definition of k -factorized is straightforward), such upper bound gets better as k increases.

If G is a group and σ is a set of primes we shall denote by G_σ a Hall subgroup of G such that $\pi(G_\sigma) \subseteq \sigma$; if $\sigma = \pi(G) \setminus \{p\}$ for a prime p , we shall write $G_{p'}$ instead of G_σ .

THEOREM A. *Let G be a group and let σ, τ, ν be three subsets of $\pi(G)$ such that $\sigma \cup \tau = \tau \cup \nu = \nu \cup \sigma = \pi$. Then*

$$h(G) \leq h(G_\sigma) + h(G_\tau) + h(G_\nu) - 2$$

In particular, if $p, q \in \pi(G)$ and $p \neq q$, then

$$h(G) \leq h(G_{p'}) + h(G_{q'}) + h(G_{\{p,q\}}) - 2.$$

THEOREM B. *Let G be a group such that $w \geq 4$. Then there exist $p, q \in \pi(G)$ with $p \neq q$ such that*

$$h(G) \leq h(G_{p'}) + h(G_{q'}) - 1.$$

A simple example will show that Theorem B does not hold when $w = 3$.
If G is a group we define

$$\mathfrak{h}_\ell(G) = \max \{ h(G_\sigma) \mid \sigma \subseteq \pi(G), |\sigma| = \ell \}$$

and we shall write \mathfrak{h}_ℓ instead of $\mathfrak{h}_\ell(G)$ if there is no possible ambiguity.

THEOREM C. *Let G be a group such that $w \geq 3$ and assume $3 \leq \ell \leq w$. Then*

$$\mathfrak{h}_\ell \leq \frac{\ell \cdot \mathfrak{h}_{\ell-1} - 2}{\ell - 2}.$$

From Theorem C we deduce

$$h(G) \leq \frac{w \cdot \mathfrak{h}_{w-1} - 2}{w - 2}.$$

In particular, if $w = 3$, then $h(G) \leq 3 \cdot \mathfrak{h}_2 - 2$ while if $w \geq 4$ then we have $h(G) \leq 2 \cdot \mathfrak{h}_{w-1} - 1$. The last bound is comparable with the one obtained in Theorem B.

§2. Notation and preliminary results.

As already pointed out, G will always denote a finite soluble group and $\{G_p\}_{p \in \pi}$ a Sylow system for G , namely a set of Sylow subgroups of G , one for any $p \in \pi$, such that $G_p G_q = G_q G_p$ for every $p, q \in \pi$.

If $\sigma \subseteq \pi$ by σ -Hall subgroup of G , we mean $G_\sigma = \prod_{p \in \sigma} G_p$; by $G_{p'}$ we denote the $\pi \setminus \{p\}$ -Hall subgroup of G . If $p \notin \pi$, by definition we set, $G_{p'} = G$; if $\pi \subseteq \sigma$ then $G_\sigma = G$ and if $\pi \cap \sigma = \emptyset$ then $G_\sigma = 1$.

The rest of the notation will be mostly standard. In particular, if p is a prime, $O_p(G)$ is the largest normal p -subgroup of G . We can write the Fitting subgroup $F(G)$ of G as $F(G) = \prod_{p \in \pi} O_p(G)$.

We shall often make use of the following result (Lemma I.7.3 of [3]; see also Lemma 1.2 of [6]).

LEMMA 2.1. ([3]) *Let G be a group and let $p \in \sigma \subseteq \pi$. Then for any $q \in \sigma$ such that $q \neq p$ we have*

$$O_p(G_\sigma) \leq O_{q'}(G).$$

Using Lemma 2.1 we can deduce Lemma 1.3 of [6], which we will prove in a way that allows us to get a better result.

PROPOSITION 2.2. *Let G be a group and $p, q \in \pi$ with $p \neq q$. If $h(G_{p'}) \leq s$, $h(G_{q'}) \leq s$ and $h(G_{\{p,q\}}) \leq r$, then $h(G) \leq s(r+1)$.*

PROOF. We use induction on r . If $r = 0$ the statement is obvious. So, assume $r \geq 1$. By Lemma 2.1 we have

$$F(G_{\{p,q\}}) = O_p(G_{\{p,q\}}) \times O_q(G_{\{p,q\}}) \leq O_{p'}(G)O_{q'}(G).$$

Let $\overline{G} = G/(O_{p'}(G)O_{q'}(G))$, then we have $h(\overline{G}_{\{p,q\}}) \leq r-1$ and, considering that $h(O_{p'}(G)O_{q'}(G)) \leq s$, the conclusion follows. \square

We shall not make use of Proposition 2.2 in what follows. It is presented here only to show how our Theorem A improves every upper bound of this kind previously known in the literature.

In order to state our results we need the following

DEFINITION 2.3. Let G be a group, $t \geq 3$ be an integer and let \mathcal{R} be the set

$$\mathcal{R} = \{\varrho_1, \varrho_2, \dots, \varrho_t \mid \varrho_i \subseteq \pi\}.$$

Then \mathcal{R} is called a *cover* of order t of π , shortly a t -cover, if $\varrho_i \cup \varrho_j = \pi$ for every $i, j \in \{1, 2, \dots, t\}$, $i \neq j$. A t -cover is called *degenerate* if $\pi \in \mathcal{R}$.

The *weight* of a t -cover $\mathcal{R} = \{\varrho_1, \varrho_2, \dots, \varrho_t\}$ is the number

$$\Theta(\mathcal{R}) = \sum_{i=1}^t h(G_{\varrho_i}).$$

When no ambiguity is possible, we shall speak simply about cover and we shall write Θ for $\Theta(\mathcal{R})$.

REMARK 2.4 Let G be a group, \mathcal{R} a t -cover of π and $w = w(G)$. Then

- (a) For every $p \in \pi$ there exists at most one $i \in \{1, 2, \dots, t\}$ such that $p \notin \varrho_i$.
- (b) The following inequality is always satisfied

$$\sum_{i=1}^t |\varrho_i| \geq (t-1) \cdot w. \quad (1)$$

- (c) Suppose that \mathcal{R} is a non degenerate t -cover of π . Then, by (1) it follows that $t = |\mathcal{R}| \leq w$. Moreover, if $|\mathcal{R}| = w$, we have $|\varrho_i| = w-1$ for every $i \in \{1, 2, \dots, t\}$, namely for every ϱ_i there exists $p_i \in \pi$ such that $G_{\varrho_i} = G_{p_i'}$.
- (d) If $\pi = \{p, q\}$, then π admits a unique 3-cover which is necessarily degenerate, namely $\mathcal{R} = \{\{p\}, \{q\}, \pi\}$.

REMARK 2.5 Let G be a group such that $w \geq 3$ and let \mathcal{R} be a t -cover of π . Let $N \trianglelefteq G$ and $\varrho_i \in \mathcal{R}$; if $\overline{G} = G/N$, then we can have $\overline{G}_{\varrho_i} = \overline{G}$. In the proofs by induction on the order of G sometimes we shall omit to mention explicitly this case. The reason for this is that we are interested in establishing an upper bound for $h(G)$ as a function of the values of $h(G_{\varrho_i})$. When $\overline{G} = \overline{G}_{\varrho_i}$ we have $h(\overline{G}) = h(\overline{G}_{\varrho_i})$ and the bound is automatically satisfied.

Similarly, if $\overline{G}_{\varrho_i} = 1$, then there exists an index j such that $\overline{G}_{\varrho_j} = \overline{G}$ and we are reduced to the case considered before.

§3. Proofs, examples and final remarks.

The following proposition is essential in order to get our results. The statement holds also when $w = 2$ by Remark 2.4 (d); however, for our purposes, it is convenient to assume the hypothesis that $w \geq 3$.

PROPOSITION 3.1 *Let G be a group such that $w \geq 3$ and let \mathcal{R} be t -cover of π of weight Θ . If $t \geq 3$, then*

$$\boxed{h(G) \leq \frac{\Theta - 2}{t - 2}} \quad (2)$$

PROOF. Let $\mathcal{R} = \{\varrho_1, \varrho_2, \dots, \varrho_t\}$, $h_i = h(G_{\varrho_i})$. We prove the claim by induction on $t + \Theta + h(G) + |G|$.

If G is nilpotent, then $\Theta = t$ and the result follows immediately. Hence we may assume that $h(G) \geq 2$. Let $F = F(G)$ be the Fitting subgroup of G . We consider two cases:

(A) $w(F) \geq 2$.

Then there exist $p, q \in \pi$ such that $O_p(G) \neq 1 \neq O_q(G)$. If neither $O_p(G)$ nor $O_q(G)$ are Sylow subgroups of G , or if $w(G) \geq 4$, then we set $\overline{G} = G/O_p(G)$ and $\widehat{G} = G/O_q(G)$. In \overline{G} and \widehat{G} the hypotheses are preserved and a standard argument gives the conclusion.

Hence suppose that $O_p(G)$ is a Sylow p -subgroup of G (clearly the roles of p and q are interchangeable) and that $w(G) = 3$. Then $\pi(G) = \{p, q, r\}$ where p, q and r are different primes, and $t - 2 = 1$. If $\pi \in \mathcal{R}$, then the statement follows immediately. Therefore we may assume that \mathcal{R} is not degenerate and $\mathcal{R} = \{\{p, q\}, \{q, r\}, \{r, p\}\}$. Note that $G/O_p(G) \simeq G_q G_r$. If $h(G_p G_q) = 1 = h(G_p G_r)$, then $G = G_p \times G_q G_r$ and $h(G) = h(G_q G_r) = h(G_p G_q) + h(G_q G_r) + h(G_r G_p) - 2$. If $h(G_p G_q) + h(G_p G_r) \geq 3$, then

$$h(G) \leq h(G_q G_r) + 1 \leq h(G_p G_q) + h(G_q G_r) + h(G_r G_p) - 2.$$

(B) $w(F) = 1$ and $F = O_p(G)$ for some $p \in \pi$.

If F is a Sylow p -subgroup of G we can get the conclusion using Remark 2.5 and arguing as in (A). Hence we may assume that $\pi(G/F) = \pi$.

By Remark 2.4.(a), possibly reordering the indices, we may assume that $F \leq G_{\varrho_i}$ for $i = 1, 2, \dots, t-1$. Moreover, since $C_G(F) \leq F$ and $C_{G_{\varrho_i}}(F(G_{\varrho_i})) \leq F(G_{\varrho_i})$, we deduce that $F(G_{\varrho_i})$ is a p -group containing F , namely $F(G_{\varrho_i}) = O_p(G_{\varrho_i})$ for every $i = 1, 2, \dots, t-1$.

From Lemma 2.1 it follows that $O_p(G_{\varrho_i}) \leq O_{q'_i}(G)$ for every $q_i \in \varrho_i \setminus \{p\}$.

Let us fix $q_1 \notin \varrho_1$; by Remark 2.4.(a) we get $q_1 \in \varrho_i$ per $i = 2, 3, \dots, t-1$. In $\overline{G} = G/O_{q'_1}(G)$ we have $h(\overline{G}_{\varrho_i}) \leq h_i - 1$ for every $i = 2, 3, \dots, t-1$ and, by inductive hypothesis

$$h(\overline{G}) \leq \frac{\left(\sum_{i=1}^t h(\overline{G}_{\varrho_i})\right) - 2}{t-2} \leq \frac{(\Theta - t + 2) - 2}{t-2} = \frac{\Theta - 2}{t-2} - 1.$$

Similarly, if $q_2 \notin \varrho_2$, then $q_2 \in \varrho_i$ for $i = 1, 3, \dots, t-1$. In $\tilde{G} = G/O_{q'_2}(G)$ we have $h(\tilde{G}_i) \leq h_i - 1$ for $i = 1, 3, \dots, t-1$ and, arguing as before, we obtain

$$h(\tilde{G}) \leq \frac{\Theta - 2}{t-2} - 1.$$

By hypothesis $t - 1 \geq 2$, hence

$$\left(\bigcap_{r \notin \{p, q_1\}} O_{r'}(G)\right) \cap \left(\bigcap_{r \notin \{p, q_2\}} O_{r'}(G)\right) = \bigcap_{r \neq p} O_{r'}(G) = O_p(G),$$

therefore

$$h(G) - 1 = h(G/O_p(G)) \leq \frac{\Theta - 2}{t-2} - 1,$$

and the conclusion follows. \square

If $t = 3$ from Proposition 3.1 it follows that $h(G) \leq \Theta - 2$; if G is nilpotent such a bound is obviously the best. We give some examples showing that the bound given by (2) is sufficiently accurate for any value of $h(G)$.

In the Examples 3.2, 3.3, 3.4 and 3.5 we shall make use of the following notation. If H is a group and $\ell \geq 1$, we shall write $[H]_\ell$ for the iterated wreath

product of H , namely $[H]_1 = H$ and $[H]_{\ell+1} = [H]_\ell \wr H$. If p, q and r are different prime numbers, let us denote by P, Q, R a p -group, a q -group and an r -group respectively. If $\pi = \{p, q, r\}$, then we set $\mathcal{R} = \{\{p, q\}, \{q, r\}, \{r, p\}\}$.

EXAMPLE 3.2. Let $G = P \wr [Q \wr R]_\ell$. Then $h(G) = 2\ell + 1$, $h(G_{r'}) = h(G_{q'}) = 2$ and $h(G_{p'}) = 2\ell$, therefore $\Theta - 2 = 2\ell + 2$.

But, if $G = P \times [Q \wr R]_\ell$ then $h(G) = 2\ell$, $h(G_{r'}) = h(G_{q'}) = 1$, $h(G_{p'}) = 2\ell$ and therefore $\Theta - 2 = 2\ell$.

EXAMPLE 3.3. Let $G = ([P \wr Q]_\ell) \wr ([R \wr Q]_\ell)$. It can be seen that $h(G) = 4\ell$ whereas $h(G_{r'}) = 2\ell$, $h(G_{p'}) = 2\ell + 1$ and $h(G_{q'}) = 2$. Hence $\Theta - 2 = 4\ell + 1$.

PROOF OF THEOREM A. With the notation of Proposition 3.1, if we set $t = 3$, $\varrho_1 = \pi \setminus \{p\}$, $\varrho_2 = \pi \setminus \{q\}$ and $\varrho_3 = \{p, q\}$, then the conclusion follows. \square

For notational convenience, given the group G , in the following proofs we shall denote by \mathcal{R}^* the w -cover $\{\pi \setminus \{p\} \mid p \in \pi\}$ of $\pi(G)$.

PROOF OF THEOREM B. Let G be a group such that $w \geq 4$. We have to show that there exist $p, q \in \pi$ such that $h(G) \leq h(G_{p'}) + h(G_{q'}) - 1$.

Let $\pi = \{p_1, p_2, \dots, p_w\}$, where the indices are ordered in a way that

$$h(G_{p'_1}) \geq h(G_{p'_2}) \geq \dots \geq h(G_{p'_w}).$$

Set $p = p_1$ and $q = p_2$.

If $h(G_{p'}) = 1$ then G is nilpotent and the statement holds. If $h(G_{p'}) \geq 2$ then we cannot have $h(G_{q'}) = 1$; for, $w \geq 4$ and from Theorem A it would follow that $h(G) \leq h(G_{p'_2}) + h(G_{p'_3}) + h(G_{p'_4}) - 2 = 1$, hence $h(G) = 1$.

If we set $\lambda = h(G_{p'}) + h(G_{q'})$ then we may assume that $\lambda \geq 4$. Considering the hypothesis that $w \geq 4$, by easy considerations we get

$$w\lambda - 4 \leq 2(w - 2)(\lambda - 1). \quad (3)$$

Given the cover \mathcal{R}^* , we certainly have

$$\Theta^* \leq \frac{w}{2} \cdot (h(G_{p'}) + h(G_{q'})) = \frac{w\lambda}{2}.$$

Therefore, by applying the inequality (3) and Proposition 3.1, we get

$$h(G) \leq \frac{\Theta^* - 2}{w - 2} \leq \frac{w\lambda - 4}{2(w - 2)} \leq \lambda - 1 = h(G_{p'}) + h(G_{q'}) - 1,$$

and the statement is proved. \square

It is not possible to extend Theorem B to the case $w = 3$; for, there exist the following

EXAMPLE 3.4. Let $G = [P \wr Q]_\ell \wr [R \wr P]_\ell \wr [Q \wr R]_\ell$ where the order of the wreath products can be chosen as you prefer. We have $h(G_{p'}) = h(G_{q'}) = h(G_{r'}) = 2\ell + 2$, while $h(G) = 6\ell$.

EXAMPLE 3.5. Let s be a prime, $s \notin \{p, q, r\}$ and let S be a non-trivial s -group. Let

$$G = [P \wr Q]_\ell \wr [P \wr R]_\ell \wr [P \wr S]_\ell \wr [Q \wr S]_\ell \wr [Q \wr P]_\ell \wr [Q \wr R]_\ell$$

where the order of the wreath products can be chosen as you prefer. We have $h(G) = 12\ell$, $h(G_{p'}) = 4\ell + 3$, $h(G_{q'}) = 4\ell + 2$ and $h(G_{\{p,q\}}) = 4\ell + 2$ hence $h(G_{p'}) + h(G_{q'}) + h(G_{\{p,q\}}) - 2 = 12\ell + 5$. Finally

$$\frac{\Theta(\mathcal{R}^*) - 2}{4 - 2} = \frac{1}{2}(h(G_{p'}) + h(G_{q'}) + h(G_{r'}) + h(G_{s'}) - 2) = 12\ell + 2.$$

PROOF OF THEOREM C. Let G be a group such that $w \geq 3$ and let $\ell \in \mathbb{N}$ where $3 \leq \ell \leq w$. We show that

$$\mathfrak{h}_\ell \leq \frac{\ell \cdot \mathfrak{h}_{\ell-1} - 2}{\ell - 2}. \quad (4)$$

Let H be a Hall subgroup of G such that $w(H) = \ell$ and $h(H) = \mathfrak{h}_\ell$. In order to prove the statement it is not restrictive to assume $G = H$. Given the cover \mathcal{R}^* , then, for every $\varrho \in \mathcal{R}^*$, we have $h(G_\varrho) \leq \mathfrak{h}_{\ell-1}$ and therefore $\Theta^* \leq \ell \cdot \mathfrak{h}_{\ell-1}$. The conclusion follows by Proposition 3.1. \square

REMARK 3.6. Let G be a group such that $w \geq 3$. Using induction and applying the inequality (4), we obtain

$$h(G) = \mathfrak{h}_w \leq \frac{w(w-1)}{2} \cdot (\mathfrak{h}_2 - 1) + 1.$$

We observe that it seems quite difficult to obtain our results omitting the hypothesis that the subgroups considered are Hall subgroups. In this direction we make the following

CONJECTURE 3.7. Let G be a group and suppose that $H, K, L \leq G$ are subgroups of G such that $G = HK = KL = LH$. Then

$$h(G) \leq h(H) + h(K) + h(L) - 2 \quad (5)$$

or, at least, there exists a function η such that $h(G) \leq \eta(h(H), h(K), h(L))$.

The unique case we know where the inequality (5) holds is when $h(H) = h(K) = 1$. For, if H, K and L are nilpotent, then a result by Kegel ([5], see also Corollary 2.5.11 of [1]), states that G is also nilpotent. The general case is covered by an unpublished result by Peterson (see Theorem 2.5.10 of [1]). Probably, in order to obtain some result in this new direction, it is necessary to use hypotheses stronger than the ones considered in Conjecture 3.7. Hence we also make the following

CONJECTURE 3.8. Let G be a group and suppose that N_1, N_2, N_3 are nilpotent subgroups of G such that $G = N_1N_2N_3$ and $N_iN_j = N_jN_i$ for every $i, j \in \{1, 2, 3\}$. Then

$$h(G) \leq h(N_1N_2) + h(N_2N_3) + h(N_3N_1) - 2.$$

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